Notes on Plücker Coordinates

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These notes on Plücker coordinates are adapted from "Geometrical Methods in Robotics" by J.M. Selig and the website `http://www.loria.fr/~lazard/ARC-Visi3D/Pant-project/files/plucker.html`. Note that specific details of the definition of Plücker coordinates varies between these two sources, and this document uses yet another definition. The definition used here is very similar to that found in Selig, but extended to work with homogeneos coordinates.

Plücker coordinates are a representation of lines in $\mathbb{R}^3$. The Plücker coordinates are unique up to a scaling. Let $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{R}^3$. In homogeneous form they are $\bar{\boldsymbol{q}}_1 = \begin{bmatrix} w_1 & x_1 & y_1 & z_1 \end{bmatrix}$ $\bar{\boldsymbol{q}}_2 = \begin{bmatrix} w_2 & x_2 & y_2 & z_2 \end{bmatrix}$, where $w_1 = w_2 = 1$. (w_i is left in these equations for generality. It will be used later.) The line containing $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ has Plücker coordinates $\begin{bmatrix} p_{01} & p_{02} & p_{03} & p_{23} & p_{31} & p_{12} \end{bmatrix}$, given by

$$\begin{aligned} p_{01} &= w_1 x_2 - w_2 x_1 & \qquad p_{23} &= y_1 z_2 - z_1 y_2 \\ p_{02} &= w_1 y_2 - w_2 y_1 & p_{31} &= z_1 x_2 - x_1 z_2 \\ p_{03} &= w_1 z_2 - w_2 z_1 & p_{12} &= x_1 y_2 - y_1 x_2 \end{aligned} \tag{1}$$

Note that with $w_1 = w_2 = 1$, the coefficients p_{01}, p_{02}, and p_{03} simplify to

$$\begin{aligned} p_{01} &= x_2 - x_1 \\ p_{02} &= y_2 - y_1 \\ p_{03} &= z_2 - z_1. \end{aligned} \tag{2}$$

The Plücker coordinate p_{ij} can be obtain by taking the determinant of the 2×2 submatrix formed by columns i and j of

$$\begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \end{bmatrix}$$

(where columns are numbers 0,1,2,3).

The direction of the line is given by

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{q}_2 - \boldsymbol{q}_1 \\ &= \begin{bmatrix} p_{01} & p_{02} & p_{03} \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{3}$$

and the moment of the line is given by

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{q}_1 \times \boldsymbol{q}_2 \\ &= \begin{bmatrix} p_{23} & p_{31} & p_{12} \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{4}$$

(The moment of a line is a vector that is normal to the plane containing the line and the origin. If the moment is 0, then the line contains the origin.)

If two different points were chosen on the line, e.g.

$$\boldsymbol{q}_1' = \lambda \boldsymbol{q}_1 + (1-\lambda)\boldsymbol{q}_2, \qquad \boldsymbol{q}_2' = \mu \boldsymbol{q}_1 + (1-\mu)\boldsymbol{q}_2.$$

Then $\boldsymbol{\omega}$ and $\boldsymbol{v}$ change only by a common scale factor:

$$\begin{aligned} \boldsymbol{\omega}' &= \boldsymbol{q}_2' - \boldsymbol{q}_1' = (\lambda - \mu)\boldsymbol{\omega} \\ \boldsymbol{v}' &= \boldsymbol{q}_1' \times \boldsymbol{q}_2' = (\lambda - \mu)\boldsymbol{v} \end{aligned}$$

Thus the Plücker coordinates are unique up to scaling.

Consider two nonparallel planes parameterized by $\boldsymbol{c}^1$ and $\boldsymbol{c}^2$, where $\boldsymbol{c}^i = \begin{bmatrix} c^i_w & c^i_x & c^i_y & c^i_z \end{bmatrix}^{\mathrm{T}}$. That is, plane i is the set of points satisfying

$$\boldsymbol{c}^{i\mathrm{T}} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = 0.$$

The two planes intersect at a line, and the Plücker coordinates for this line can be calculated using a duality priciple. Treat $\boldsymbol{c}^1$ and $\boldsymbol{c}^2$ as points in homogeneous form and calculate the Plücker coordinates $\begin{bmatrix} p_{01} & p_{02} & p_{03} & p_{23} & p_{31} & p_{12} \end{bmatrix}$ using Eqn 1. The Plücker coordinates for the line defined by the intersection of the two planes are obtained by swapping the two triplets, i.e. $\begin{bmatrix} p_{23} & p_{31} & p_{12} & p_{01} & p_{02} & p_{03} \end{bmatrix}$.

Given the Plücker coordinates of a line, the points that lie on the line must satisfy

$$\begin{aligned} \mathbf{0} &= \boldsymbol{\omega} \times \boldsymbol{q} - \boldsymbol{v} \\ &= \hat{\boldsymbol{\omega}} \boldsymbol{q} - \boldsymbol{v} \\ &= \begin{bmatrix} \boldsymbol{v} & \hat{\boldsymbol{\omega}} \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{q} \end{bmatrix} \end{aligned} \tag{5}$$

where

$$\hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{6}$$

This system will have rank 2, and thus defines a line in $\mathbb{R}^3$. Points on the line also must satisfy $\boldsymbol{v}^{\mathrm{T}} \boldsymbol{q} = \mathbf{0}$. These conditions can be combined into the following matrix equation

$$\begin{bmatrix} 0 & -p_{23} & -p_{31} & -p_{12} \\ p_{23} & 0 & -p_{03} & p_{02} \\ p_{31} & p_{03} & 0 & -p_{01} \\ p_{12} & -p_{02} & p_{01} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{q} \end{bmatrix} = \mathbf{0}$$

This matrix also will have rank 2. Each line defines a plane which intersects the line.